Lecture 7: Derivatives & Integrals of VVF

Limits of VVF

We define the limit of a VVF, $\bar{F}(t)$, as:

$$\lim_{t \to t_0} \bar{F}(t) = \bar{L}$$

and say this limit exists iff $\forall \varepsilon > 0 \ \exists \ \delta > 0$ s.t.

if $0 < |t - t_0| < \delta$, then $\|\bar{F}(t) - L\| < \varepsilon$.

Intuitively this means that we can pick a value of t that makes $\bar{F}(t)$ as close to $\bar{L}$ as we wish.

If $\bar{F}(t) = \langle f_1(t), f_2(t), f_3(t) \rangle$ then the limit of $\bar{F}(t)$ at t_0 exists iff $f_1, f_2, \& f_3$ also have limits at t_0. Then

$$\lim_{t \to t_0} \bar{F}(t) = \langle \lim_{t \to t_0}(f_1(t)), \lim_{t \to t_0}(f_2(t)), \lim_{t \to t_0}(f_3(t)) \rangle$$

Derivative of VVF

If the limit,

$$\lim_{t \to t_0} \frac{\bar{F}(t) - \bar{F}(t_0)}{t - t_0},$$

exist we call it the derivative of $\bar{F}(t)$ at t_0. We write

$$\frac{d\bar{F}(t)}{dt} = \bar{F}'(t) = \lim_{t \to t_0} \frac{\bar{F}(t) - \bar{F}(t_0)}{t - t_0}$$

In terms of components: Given $\bar{F}(t) = \langle f_1(t), f_2(t), f_3(t) \rangle$

$$\bar{F}'(t) = \langle f_1'(t), f_2'(t), f_3'(t) \rangle$$

Geometrically, the vector $\bar{F}'(t)$ is a tangent vector. Specifically, $\bar{F}(t_0)$, is a vector tangent to $\bar{F}(t)$ at the point $\bar{F}(t_0)$.

Ex. 1 Find $\frac{d\bar{F}}{dt}$ given $\bar{F}(t) = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$.

$f_1' = a, \quad f_2' = b, \quad f_3' = c$

$\frac{d\bar{F}}{dt} = \langle a, b, c \rangle$ a constant vector.

Ex. 2 Find a unit tangent vector to $\bar{r}(t) = \langle 1 + t^3, te^{-t}, \sin(2t) \rangle$ @ $t = \emptyset$

$f_1' = 3t^2, \quad f_2' = e^{-t} + -te^{-t} \quad f_3' = 2\cos(2t)$

$\bar{r}'(t) = \langle 3t^2, (1-t)e^{-t}, 2\cos(2t) \rangle$

$\bar{r}'(t = \emptyset) = \langle 3(\emptyset)^2, (1-\emptyset)e^{-0}, 2\cos(2(\emptyset)) \rangle$

$\bar{r}'(t = \emptyset) = \langle \emptyset, 1, 2 \rangle$

unit tangent $= \left.\frac{\bar{r}'}{|\bar{r}'|}\right|_{t=\emptyset} = \frac{\langle \emptyset, 1, 2 \rangle}{\sqrt{5}} = \langle \emptyset, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle$

Identities

$(\bar{F} + \bar{G})' = \bar{F}' + \bar{G}'$

$(\bar{F} - \bar{G})' = \bar{F}' - \bar{G}'$

$(f\bar{F})' = f'\bar{F} + f\bar{F}'$

$(\bar{F} \cdot \bar{G})' = \bar{F}' \cdot \bar{G} + \bar{F} \cdot \bar{G}'$

$(\bar{F} \times \bar{G})' = \bar{F}' \times \bar{G} + \bar{F} \times \bar{G}'$

$(\bar{F} \circ g)' = \bar{F}'(g)g'$

Derivatives of Position

In physics we often discuss the position vector $\bar{r}(t) = \langle x(t), y(t), z(t) \rangle$. The derivatives of $\bar{r}(t)$ are also very important.

Position: $\bar{r}(t) = \langle x(t), y(t), z(t) \rangle$

Velocity: $\bar{v}(t) = \frac{d\bar{r}}{dt} = \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle$

Speed: $\|\bar{v}(t)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$

Acceleration: $\bar{a}(t) = \frac{d\bar{v}}{dt} = \frac{d^2\bar{r}}{dt^2} = \langle \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt} \rangle$

Integrals of VVF

If $\bar{F}(t)$ is continuous in an interval $[a,b]$ we can define the integral:

$$\int_a^b \bar{F}(t)\,dt = \lim_{n\to\infty} \sum_{i=1}^{n} \bar{F}(t_i^*)\Delta t, \quad t_i^* \text{ is a "sample point" in the } i^{th} \text{ interval}$$

$$= \lim_{n\to\infty} \left[\left(\sum_{i=1}^{n} f_1(t_i^*)\Delta t\right)\hat{i} + \left(\sum_{i=1}^{n} f_2(t_i^*)\Delta t\right)\hat{j} + \left(\sum_{i=1}^{n} f_3(t_i^*)\Delta t\right)\hat{k} \right]$$

We can instead write it as:

$$\int_a^b \bar{F}(t)\,dt = \left< \int_a^b f_1(t)\,dt, \int_a^b f_2(t)\,dt, \int f_3(t)\,dt \right>$$

The integral of a VVF can be found by finding the integrals of the component functions. This also applies to indefinite integrals.

$$\int \bar{F}(t)\,dt = \left< \int f_1(t)\,dt, \int f_2(t)\,dt, \int f_3(t)\,dt \right>$$

Ex. 3 An object with initial position, r_0, initial velocity, V_0, is in free fall. Find $\bar{r}(t)$.

$$\bar{a}(t) = -g\hat{k}$$

$$\bar{v}(t) = \int \bar{a}(t)\,dt = \int -g\hat{k}\,dt = -gt\,\hat{k} + \bar{C}$$

$$\bar{v}_0 = -g(0)\hat{k} + \bar{C} = \bar{V}_0 \rightarrow \bar{C} = \bar{V}_0$$

$$\bar{r}(t) = \int \bar{v}(t)\,dt = \int -gt\,dt\,\hat{k} + \int \bar{V}_0\,dt = -\tfrac{1}{2}gt^2\hat{k} + \bar{V}_0 t + \bar{C}_1$$

$$\bar{r}_0 = -\tfrac{1}{2}g(0)^2\hat{k} + \bar{V}_0(0) + \bar{C}_1 = \bar{r}_0 \rightarrow \bar{C}_1 = \bar{r}_0$$

$$\bar{r}(t) = \bar{r}_0 + \bar{V}_0 t - \tfrac{1}{2}gt^2\hat{k}$$